AN APPLICABLE REPRESENTATION OF QUADRATIC FORM TO SHOW REPRESENTATION OF m-GONAL FORM

Abstract. The *m-gonal number*

$$P_m(x)=\frac{m-2}{2}(x^2-x)+x \tag{0.1}$$

where $x\in\mathbb{N}$ which is defined as the total number of dots to constitute the m-gon (with x dots for each side) has been popular subject from 2nd Century in B.C in the research of number theory. Fermat famously claimed that every positive integer is written as at most m m-gonal numbers. Lagrange and Gauss resolved his claim for $m=4$ and $m=3$, respectively and finally, Cauchy completed its proof for all m.

On the other hand, by admitting zero and negative integer too for x in (0.1), we may generalize the m-gonal number. We call a weighted sum of m-gonal numbers

$$F_m(\mathbf{x})=a_1P_m(x_1)+\cdots+a_nP_m(x_n)=:\langle a_1,\cdots,a_n\rangle_m \tag{0.2}$$

where $a_i\in\mathbb{N}$ an *m-gonal form.* If the diophantine equation $F_m(\mathbf{x})=N$ has an integer solution $\mathbf{x}\in\mathbb{Z}$ for some $N\in\mathbb{N}$, we say that $F_m(\mathbf{x})$ *represents* N and if $F_m(\mathbf{x})$ represents every positive integer, then we say that $F_m(\mathbf{x})$ is *universal.*

Since the smallest m-gonal number is $m-3$ except 0 and 1, in constructing an universal m-gonal form, escalating process to represent from 1 to at least $m-4$ must occur. But the representability of every positive integer up to $m-4$ does not characterize the universality i.e., there are m-gonal forms which represent every positive integer up to $m-4$ but not universal. Then one may question that how far is it to an m-gonal form which represents every positive integer up to $m-4$ from an universal form? Firstly, we show that for any m-gonal form $F_m(\mathbf{x})=\sum_{i=1}^{n}a_iP_m(x_i)$ which represent every positive integer up to $m-4$, there is an $a_{n+1}\in\mathbb{Z}$ for which $F_m(\mathbf{x})+a_{n+1}P_m(x_{n+1})$ is universal for $m\geq 12$.

Even though the representability of every positive integer up to $m-4$ is not enough to imply the universality in general, for some specific pairs of first coefficients, the representability of m-gonal form whose first coefficients are specific (kinds of very nice) is equivalent its universality. Secondly, we classify such the coefficients (a_1,a_2,a_3,a_4,a_5) and see some applications to determine the optimal (i.e., minimal) rank of some specific type of m-gonal forms to be univesal.

In order to show both of above two arguments, we adopt the arithmetic theory of quadratic form. Especially, the proof relies on observation of particular representation of specific type of binry quadratic form by a diagonal quadratic form.